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THE STABILITY OF THE CYLINDRICAL PRECESSION OF A VISCOELASTIC BODY UNDERGOING OSCILLATIONS ALONG AN AXIS OF SYMMETRY†

YE. V. SINITSYN

Moscow

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The relative motion of a viscoelastic body undergoing longitudinal oscillations along an axis of symmetry in a Newtonian central gravitational field is considered. The linear theory of viscoelasticity is used. The stability of a particular solution of the equations of motion is studied. This solution corresponds to the uniform rotation of the body about an axis of symmetry perpendicular to the plane of the circular orbit. Using the reduction principle of the theory of denumerable systems of differential equations and Kamenkov's criterion, the stability in the whole domain of variation of the parameters in the non-linear problem is investigated. It is shown that the "quasi-static approach" to studying stability is correct.

1. FORMULATION OF THE PROBLEM. THE EQUATIONS OF MOTION

CONSIDER a homogeneous dynamically symmetric viscoelastic isotropic body moving in a Newtonian central field of force. We will assume that the centre of mass moves around a fixed circular orbit ($\omega_0 = 2\pi T^{-1}$, where T is the period of revolution of the centre of mass around the orbit).

With the deformable body we connect a central coordinate system $Ox_1x_2x_3$ with origin at the centre of mass (see, for example, [1–3]). The Ox_3 axis is parallel to the axis of symmetry.

We introduce the following notation: $\mathbf{r}$ is the radius vector of particle dm of the body in the non-deformed state relative to the central system of axes, and $\mathbf{u}(\mathbf{r}, t)$ is the elastic displacement of the particle. We shall use the expansion

$$\mathbf{u}(\mathbf{r},t) = \sum_{n=1}^{\infty} q_n(t)\mathbf{U}_n(\mathbf{r})$$

of $\mathbf{u}(\mathbf{r}, t)$ in an orthonormal system of characteristic modes of elastic oscillations of the body, $q_n(t)$ being generalized (normal) coordinates. We will assume that the body undergoes only longitudinal oscillations parallel to the axis of symmetry. The displacement $\mathbf{u}(\mathbf{r}, t)$ is then parallel to the Ox_3 axis and $\mathbf{U}_n = (0, 0, U_n(x_3))^T$.

As an example one can consider the case when the body is elongated in the direction of the axis of symmetry and the oscillation modes can be well approximated by the characteristic modes of longitudinal elastic oscillations of a rod [1, 4].

We shall assume that the material of the body satisfies the Kelvin–Voight model of linear

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theory of viscoelasticity with constant Poisson's ratio (independent of time). The potential energy of elastic strains can be written in the form

$$\Pi_* = \frac{1}{2}\sum_{n=1}^{\infty}\Omega_n^2 q_n^2$$

Here Ω_n are the characteristic frequencies of free elastic oscillations.

The dissipative functional can be represented in the form of the Rayleigh functional

$$\Psi = \chi b \sum_{n=1}^{\infty}\Omega_n^2 \dot{q}_n^2$$

where $b = \text{const}$ and χ is a dimensionless coefficient. Here and throughout this paper the time derivative of a scalar or vector in the central coordinate system is denoted by a dot.

We introduce the orbital coordinate system $OX_1X_2X_3$. The axis OX_3 is parallel to the radius vector of the centre of mass, while the axes OX_2 and OX_1 are parallel to the binormal direction to the orbit and the transversal line pointing in the direction of motion of the centre of mass, respectively.

Let ω be the absolute angular velocity of the trihedron $Ox_1x_2x_3$, and let γ be the unit vector of the axis OX_3 (ω_i and γ_i are the projections of the vectors ω and γ on the Ox_i axes, respectively). We shall define the orientation of the trihedron $Ox_1x_2x_3$ relative to the orbital coordinate system by means of the Euler angles φ, ψ, θ introduced in the usual way.

The differential equations describing the motion of the trihedron $Ox_1x_2x_3$ and the deformation of the body can be written in the form [2]

$$(J\boldsymbol{\omega})^{\cdot} + \boldsymbol{\omega}\times J\boldsymbol{\omega} = 3\omega_0^2\boldsymbol{\gamma}\times J\boldsymbol{\gamma} \tag{1.1}$$

$$\ddot{q}_n + 2\chi b\Omega_n^2\dot{q}_n + \Omega_n^2 q_n = Q_n, \quad n = 1,2,\ldots \tag{1.2}$$

$$Q_n = (H_n + q_n)(\omega_1^2 + \omega_2^2 + \omega_0^2(3\gamma_3^2 - 1))$$

In the case under consideration, in which the body is dynamically symmetric and undergoes longitudinal oscillations along the axis of symmetry, the tensor of the inertia J relative to O can be represented in the form

$$J = J_0 + J_1 + J_2 = \operatorname{diag}\{A, A, C\} \tag{1.3}$$

$$J_0 = \operatorname{diag}\{A_0, A_0, C_0\}, \quad J_1 = 2\sum_{n=1}^{\infty} q_n J_1^{(n)}, \quad J_2 = \sum_{n=1}^{\infty} q_n^2 J_2^{(n)}$$

$$J_1^{(n)} = \operatorname{diag}\{H_n, H_n, 0\}, \quad J_2^{(n)} = \operatorname{diag}\{1,1,0\}$$

$$H_n = \int_\Omega x_3 U_n \, dm$$

where the integral is computed over the region occupied by the body in the non-deformed state.

We can see that the axes Ox_i ($i = 1, 2, 3$) remain the main central axes of inertia during the motion. The moments of inertia relative to these axes depend on time via the coordinates q_n.

The components ω_i of the angular velocity vector and the direction cosines γ_i of the radius vector of the centre of mass can be expressed in terms of the Euler angles by the well-known formulae

$$\omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi + \omega_0(\sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta) \tag{1.4}$$

$$\omega_2 = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi - \omega_0(\sin\psi\sin\varphi - \cos\psi\cos\varphi\cos\theta)$$

$$\omega_3 = \psi^{\cdot}\cos\theta + \varphi^{\cdot} - \omega_0\cos\psi\sin\theta$$
$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

We write (1.1) in the scalar form and substitute (1.4) into the resulting equations. Then we add the first equation multiplied by sin φ to the second equation multiplied by cos φ. Next, we subtract the second equation multiplied by sin φ from the first equation multiplied by cos φ. We obtain equations of motion of the form (see also [1])

$$\begin{aligned}&\psi''\sin\theta + 2\psi'\theta'\cos\theta - 2\theta'\sin\theta\cos\psi - \cos\psi\sin\psi\sin\theta - \\ &-\frac{C}{A}\beta(\theta' + \sin\psi) + \frac{A'}{A}(\psi'\sin\theta + \cos\psi\cos\theta) = 0\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\theta'' - \psi'^2\sin\theta\cos\theta + 2\psi'\cos\psi\sin^2\theta + \cos^2\psi\sin\theta\cos\theta - \\ &-3\left(\frac{C}{A} - 1\right)\sin\theta\cos\theta + \beta\frac{C}{A}(\psi'\sin\theta + \cos\psi\cos\theta) + \frac{A'}{A}(\theta' + \sin\psi) = 0\end{aligned}$$

$$C\dot{\omega}_3 = 0 \;\Rightarrow\; \omega_3 = r_0 = \text{const}$$

$$(\beta = r_0/\omega_0, \; (\cdot)' = d(\cdot)/d\tau, \; \tau = \omega_0 t)$$

In (1.2) we also change to dimensionless time and we write the equations in the form

$$\begin{aligned}&q_n'' + 2\chi b\omega_0\omega_n^2 q_n' + \omega_n^2 q_n = (q_n + H_n)(\psi'^2\sin^2\theta + \\ &\theta'^2 + \sin^2\psi + 4\cos^2\theta + 2\psi'\cos\psi\cos\theta\sin\theta + 2\theta'\sin\psi - 1) \\ &\omega_n = \Omega_n/\omega_0, \quad n = 1,2,\ldots\end{aligned} \tag{1.6}$$

Equations (1.5) and (1.6) describe the motion of the trihedron $Ox_1x_2x_3$ and the deformations of the body. They form a closed denumerable system of ordinary differential equations. The angular coordinate φ has the character of a cyclic variable.

The system of equations (1.5), (1.6) has the exact particular solution

$$\psi = \pi, \quad \theta = \frac{\pi}{2}, \quad q_n = -\frac{H_n}{\omega_n^2 + 1} \quad (n = 1,2,\ldots) \tag{1.7}$$

This solution corresponds to the case when the dynamical axis of symmetry Ox_3 of the body is orthogonal to the plane of the orbit at any instant during the motion, while the body rotates about the axis Ox_3 with constant angular velocity $\omega_3 = r_0$. In the case of an absolutely rigid body such a motion is called cylindrical precession [5], because the axis Ox_3 traces a cylindrical surface in inertial space.

A rigorous study of the stability of the above-mentioned particular solution of the non-linear system (1.5), (1.6) relative to the variables $\psi, \psi', \theta, \theta', \varphi', q_n, q_n'$ will be carried out below.

2. THE METHOD OF INVESTIGATION

The formulation of the problem differs from that considered earlier in [1] only by the fact that bending oscillations such that the displacement vector $\mathbf{u}(\mathbf{r}, t)$ is perpendicular to the axis Ox_3 were allowed in [1]. However, such oscillations do not contribute to the linear part of the inertia tensor and do not affect the result to within the approximations used in [1].

We shall describe the method employed in [1]. It is assumed that

$$\omega_n^{-1} \sim \varepsilon << 1, \quad \chi \sim \varepsilon^{1+\delta} \quad (0 < \delta < 1)$$

The system of equations of motion is reduced to singularly perturbed equations, an asymptotic solution of which can be constructed by the boundary function method [6, 7]. The solutions for q_n corresponding to the semi-classical oscillation regime are substituted into the equations for ψ and θ, which form a closed system, and stability is studied with the aid of Kamenkov's criterion. A rigorous justification of this method will be presented below.

The justification is necessary, firstly, because the methods of the theory of singularly perturbed equations employed have been developed for systems with a finite number of degrees of freedom. The question of justifying the methods for systems in infinite-dimensional spaces remains open for the time being (here we should mention the paper by Shatina [8] in which the systems considered are similar to those under consideration, but have a more specialized form). Secondly, stability is studied using the equations satisfied by an approximate solution, which differs from the exact solution by $(\sim \varepsilon^4)$ in a large $(\sim \varepsilon^{-1})$ but finite time interval. The "semi-classical" approach from [1] will be substantiated below without using any asymptotic methods of solving differential equations.

Let A^* be the moments of inertia of the body about the axes Ox_1, Ox_2 obtained from (1.3) for $q_n = -H_n(\omega_n^2+1)^{-1}$.

We introduce the perturbations x_1, x_2, ξ_n by

$$\psi = \pi + x_1, \quad \theta = \frac{\pi}{2} + x_2, \quad q_n = -\frac{H_n}{\omega_n^2+1} + \xi_n$$

and write the equations in terms of deflections

$$\begin{aligned} &x_1'' + (2-\alpha\beta)x_2' + (\alpha\beta - 1)x_1 = G_1 \\ &x_2'' - (2-\alpha\beta)x_1' + (\alpha\beta + 3\alpha - 4)x_2 = G_2 \end{aligned} \tag{2.1}$$

$$\xi_n'' + 2\chi b\omega_0\omega_n^2\xi_n' + (\omega_n^2+1)\xi_n = Z_n + O_n^3 \tag{2.2}$$

$$Z_n = \frac{\omega_n^2 H_n}{\omega_n^2+1}(x_1^2 + 4x_2^2 + x_1'^2 + x_2'^2 + 2(x_1'x_2 - x_2'x_1))$$

where $\alpha = C_0 A^{*-1}$, and where the terms G_1, G_2 are at least of order two with respect to x_1, x_1', x_2, x_2', ξ_i, ξ_i' and O_n^3 is at least of order three. Equations (2.1) and (2.2) can be obtained from the Taylor expansions of the right-hand sides of system (1.5), (1.6) in the neighbourhood of the solution (1.7).

We will consider the system obtained from (2.1) and (2.2) by neglecting the non-linear terms. Subject to the conditions

$$\begin{gathered} (\alpha\beta - 1)(\alpha\beta + 3\alpha - 4) > 0, \quad \alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1 > 0 \\ (\alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1)^2 - 4(\alpha\beta - 1)(\alpha\beta + 3\alpha - 4) > 0 \end{gathered} \tag{2.3}$$

which are analogous to those known from the theory of the motion of a rigid body in a

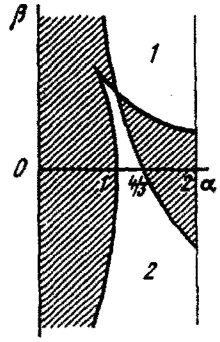

FIG. 1.

gravitational field [5], the linear approximation of system (2.1) has two pairs of purely imaginary roots $\pm i\delta_1$, $\pm i\delta_2$ $(\delta_1 > \delta_2)$. If at least one of conditions (2.3) is violated, then the characteristic equation has a root with positive real part. The domains in the α, β plane in which the conditions (2.3) are satisfied are presented in Fig. 1 (domains 1 and 2). We remark that the moments of inertia involved in the definition of α correspond to a state of the body other than the non-deformed state $(q_n = -H_n(\omega_n^2+1)^{-1})$. Henceforth system (2.1) will be called critical.

The solution of the system obtained from (2.2) by neglecting non-linear terms satisfies the inequality

$$\|z\| \leqslant \|z(t_0)\| B e^{-\tilde{\alpha}(t-t_0)} \tag{2.4}$$

for $t \geqslant t_0 > 0$, where $B \geqslant 1$ and $\tilde{\alpha} > 0$ are constants independent of the choice of $t_0 \geqslant 0$. The norm $\|z\|$ in the space of sequences $z = (\xi_1, \xi_1', \xi_2, \xi_2', \ldots)$ will be introduced below.

We shall use the reduction principle of stability theory [9] to study the stability of the zero solution of system (2.1), (2.2).

A generalization of this principle for certain cases of denumerable systems can be found in [10]. In the general case of denumerable systems the reduction principle is stated without proof in [11], where no correction is taken into account in the formulation [9, footnote on p. 383].

We introduce the notion of a "truncated" system [9] obtained from (2.1) for $\xi_n = 0$, $\xi_n' = 0$ $(n = 1, 2, \ldots)$.

In the following theorem we state the reduction principle in terms of the problem under consideration.

Theorem. We assume that the unperturbed motion $x_1 = x_1' = x_2 = x_2' = 0$ of the "truncated" system is either stable, or asymptotically stable, or unstable if the dependence on terms of order higher than N is neglected. This being the case, if the expansion of each of the functions $Z_n + O_n^3(x_1, x_1', x_2, x_2', 0, 0)$ begins with a term of order greater than or equal to $N+1$ then, for the complete system (2.1), (2.2), the unperturbed motion $x_1 = x_1' = x_2 = x_2' = \xi_n = \xi_n' = 0$ is, respectively, stable, asymptotically stable, or unstable.

The proof of this assertion will be omitted because it is tedious and trivial. It uses the transformations of the system of equations considered in [10, 11], the proof of the reduction principle in the case of a finite number of degrees of freedom [12], and the theorems of the second Lyapunov method for denumerable systems of equations [10].

Note that it follows immediately from the above theorem that the unperturbed motion is unstable in those domains in which at least one of the conditions (2.3) is violated (the hatched areas in the figure).

The theorem cannot be applied to study stability in domains 1 or 2, because the expansion of Z_n contains second-order terms. The stability of the "truncated" system corresponding to the case of two pairs of purely imaginary roots can be demonstrated using Kamenkov's criterion ($N = 3$). In the case of a system of finite order this problem can be solved using a replacement of variables of the form

$$\xi_n = z_n + \sum_{m_1+m_2+n_1+n_2=2} a^n_{m_1 m_2 n_1 n_2} x_1^{m_1} x_2^{m_2} x_1'^{n_1} x_2'^{n_2} \tag{2.5}$$

which is close to an identity.

The constants $a^n_{m_1 m_2 n_1 n_2}$ in (2.5) are chosen so as to ensure that the quadratic terms in Z_n vanish. The expansion of each of the functions $O_n^3(x_1, x_1', x_2, x_2', 0, 0)$ begins with a term of order greater than or equal to four.

In the general case of denumerable systems the problem of constructing such a replacement of variables can be reduced to the problem of solving a denumerable system of algebraic equations with respect to the system of constants $a^n_{m_1 m_2 n_1 n_2}$.

In the case of system (2.2) the equation is different for each n, which makes it possible to construct the general substitution formula (2.5) and find $a^n_{m_1 m_2 n_1 n_2}$. On substituting (2.5) into system (2.1), one must set $z_n = z_n' = 0$ (by virtue of the linear approximation equations, z_n' can be found from (2.5) by differentiation).

Then the argument concerning the stability of the new "truncated" system, the right-hand sides of which differ from those of the "truncated" system (2.1) by higher-order variables, answers the question of the stability of the original system (2.1), (2.2).

Thus the reduction principle of stability theory has been generalized to the case of denumerable systems of differential equations. The problem of the stability of the cylindrical precession of a viscoelastic body undergoing longitudinal oscillations along an axis of symmetry (Sec. 1) can serve as an example of the application of the theorem stated above.

3. INVESTIGATION OF STABILITY

We shall study the stability of unperturbed motion in those domains of variation of α and β in which conditions (2.3) are satisfied (domains 1 and 2 in Fig. 1). We shall substitute (2.5) into (2.2). On equating the coefficients multiplying the quadratic terms with respect to x_i and x_i', we obtain a system of equations from which we determine the coefficients $a^n_{m_1 m_2 n_1 n_2}$. The system has the matrix form

$$\begin{gathered}\left(B^2 + 2\chi b\omega_0\omega_n^2 B + \left(\omega_n^2+1\right)E\right)\mathbf{a} = \frac{\omega_n^2 H_n}{\omega_n^2+1}\mathbf{c} \\ \mathbf{a} = (a^n_{2000}, a^n_{1010}, a^n_{0020}, a^n_{1100}, a^n_{0110}, a^n_{0200}, a^n_{1001}, a^n_{0011}, a^n_{0101}, a^n_{0002})^T \\ \mathbf{c} = (1,0,1,0,2,4,-2,0,0,1)^T\end{gathered} \tag{3.1}$$

where B is a 10×10-matrix, which can be obtained by differentiating the expression

$$\sum_{m_1+m_2+n_1+n_2=2} a^n_{m_1 m_2 n_1 n_2} x_1^{m_1} x_2^{m_2} x_1'^{n_1} x_2'^{n_2}$$

with respect to the time variable, by virtue of the linear approximation equations of system (2.1) and collecting similar terms, where E is the unit matrix. The explicit form of B is not required in what follows.

System (3.1) is an inhomogeneous system of 10 linear algebraic equations with 10 unknowns. The general solution of this system is quite involved. We will make the following assumptions: let the period of free elastic oscillations of the body corresponding to the lowest harmonic be much shorter than the characteristic damping time of these oscillations and let these two numbers be much smaller than the period of revolution of the centre of mass around the orbit.

We set

$$\omega_n = \varepsilon^{-1}\tilde{\omega}_n, \quad \chi \sim \varepsilon^{1+\delta}(0<\delta<1), \quad 0<\varepsilon<<1 \tag{3.2}$$

We remark that the same assumptions have been used before in [1, 2, 4, 7] to construct an asymptotic solution of the equations of motion. Here the asymptotic expansions (3.2) are used to solve the system of algebraic equations.

The assumptions (3.2) are in good agreement with the assumptions that the theory of small deformations is correct and the representation of the potential energy of elastic deformations as a quadratic functional (Sec. 1).

Taking (3.2) into account, we can write (3.1) in the form

$$(\varepsilon^2 B^2 + 2\chi b\omega_0\tilde{\omega}_n^2 B + \left(\tilde{\omega}_n^2 + \varepsilon^2\right)E)\mathbf{a} = \frac{\varepsilon^2\tilde{\omega}_n^2 H_n}{\tilde{\omega}_n^2 + \varepsilon^2}\mathbf{c} \tag{3.3}$$

We will express the solution of (3.3) as a convergent series, confining ourselves to the first two terms

$$\mathbf{a} = \varepsilon^2 H_n\tilde{\omega}_n^{-2}(\mathbf{c} - 2\chi b\omega_0 B\mathbf{c}) + O(\varepsilon^4) \tag{3.4}$$

In view of (3.4), the substitution (2.5) can be represented in the form

$$\xi_n = z_n + \varepsilon^2\tilde{\omega}_n^{-2}(Z_n - 2\chi b\omega_0 Z_n') + O(\varepsilon^4) \tag{3.5}$$

Next, we substitute (3.5) into (2.1) and, setting $z_n = 0$ and $z_n' = 0$, we study the stability of the unperturbed motion of the critical system, the system under investigation being exactly the same as the system of equations of the "quasi-static approximation" [1].

It follows that the application of the "quasi-static approach" [1] to study stability has been justified mathematically.

The fact that the approximate solution (3.4) is used instead of the exact solution of (3.1) means that the boundaries of the domains of stability will be found with an error of order ε^2, higher-order terms of expansion (3.4) being necessary to improve accuracy.

On applying the transformations described above, we obtain the system of equations

$$\begin{aligned}
&x_1'' + (2-\alpha\beta)x_2' + (\alpha\beta - 1)x_1 = g^1 + O_5^1 \\
&x_2'' - (2-\alpha\beta)x_1' + (\alpha\beta + 3\alpha - 4)x_2 = g^2 + O_5^2 \\
&g^i = \sum_{m_1+m_2+n_1+n_2=3} g^i_{m_1m_2n_1n_2} x_1^{m_1} x_2^{m_2} x_1'^{n_1} x_2'^{n_2}
\end{aligned} \tag{3.6}$$

The coefficients $g^i_{m_1m_2n_1n_2}$ are listed in the tables in [1]. One of them contains an error, namely, the expression for g^0_{2100} has the wrong sign. This resulted in an incorrect picture of the stability domains in domains 1 and 2. These are corrected below.

We shall study the stability of the zeroth solution of system (3.6) using Kamenkov's criterion [12, 13].

We make the substitution

$$x_1 = -\frac{i}{2\sqrt{\delta_1}}w_1 + \frac{i}{2\sqrt{\delta_1}}w_1^* - \frac{i}{2\sqrt{\delta_2}}w_2 + \frac{i}{2\sqrt{\delta_2}}w_2^*$$

$$x_2 = \frac{\sqrt{\delta_1}}{2}w_1 + \frac{\sqrt{\delta_1}}{2}w_1^* + \frac{\sqrt{\delta_2}}{2}w_2 + \frac{\sqrt{\delta_2}}{2}w_2^*$$

$$x_1' = -\frac{k_1}{2\sqrt{\delta_1}}w_1 - \frac{k_1}{2\sqrt{\delta_1}}w_1^* - \frac{k_2}{2\sqrt{\delta_2}}w_2 - \frac{k_2}{2\sqrt{\delta_2}}w_2^* \tag{3.7}$$

$$x_2' = -\frac{ik_1\sqrt{\delta_1}}{2}w_1 + \frac{ik_1\sqrt{\delta_1}}{2}w_1^* - \frac{ik_2\sqrt{\delta_2}}{2}w_2 + \frac{ik_2\sqrt{\delta_2}}{2}w_2^*$$

$$k_j = \frac{(\alpha\beta-1)-\delta_j^2}{\delta_j(\alpha\beta-2)} = \frac{\delta_j(\alpha\beta-2)}{(\alpha\beta+3\alpha-4)-\delta_j^2}$$

Here w^*_i is the complex number conjugate to w_i. The new variables satisfy the equations

$$\begin{aligned} w_1' &= i\delta_1 w_1 + \sum_{m_1+m_2+n_1+n_2=3} A^1_{m_1m_2n_1n_2} w_1^{m_1} w_2^{m_2} w_1^{*n_1} w_2^{*n_2} \\ w_2' &= i\delta_2 w_2 + \sum_{m_1+m_2+n_1+n_2=3} A^2_{m_1m_2n_1n_2} w_1^{m_1} w_2^{m_2} w_1^{*n_1} w_2^{*n_2} \end{aligned} \tag{3.8}$$

the coefficients $A^i_{m_1m_2n_1n_2}$ being expressed in terms of $g^i_{m_1m_2n_1n_2}$ in a complex way. By Kamenkov's criterion, the zero equilibrium is asymptotically stable if the following three conditions are satisfied simultaneously [12, 13]

$$\begin{gathered} (1)\ A^1_{2010} < 0,\ (2)\ A^2_{0201} < 0,\ (3) \text{ if } A^1_{1101} > 0 \text{ and } A^2_{1110} > 0, \\ \text{then } \Delta = A^1_{2010}A^2_{0201} - A^1_{1101}A^2_{1110} > 0. \end{gathered} \tag{3.9}$$

The equilibrium is unstable if at least one of the inequalities in conditions 1–3 is (strictly) violated (in the case of condition 3, if the inequality involving Δ is violated).

In the case under consideration, after fairly lengthy computation we find that

$$\begin{aligned} A^1_{2010} &= \frac{P_1^1 k_2}{k_2\delta_1 - k_1\delta_2} + \frac{P_2^1}{k_2\delta_2 - k_1\delta_1}, \quad A^2_{0201} = -\frac{P_1^2 k_1}{k_2\delta_1 - k_1\delta_2} - \frac{P_2^2}{k_2\delta_2 - k_1\delta_1} \\ A^1_{1101} &= \frac{G_1^{12} k_2}{k_2\delta_1 - k_1\delta_2} + \frac{G_2^{12}}{k_2\delta_2 - k_1\delta_1}, \quad A^2_{1110} = -\frac{G_1^{21} k_1}{k_2\delta_1 - k_1\delta_2} - \frac{G_2^{21}}{k_2\delta_2 - k_1\delta_1} \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} P_1^i &= g^1_{0300}\left(-3k_i^3\delta_i^{-1}\right) + g^1_{0210}\left(3k_i^2\right) + g^1_{0120}\left(-3k_i\delta_i\right) + \\ &g^1_{1011}\left(k_i\delta_i\right) + g^1_{0102}\left(-3k_i^3\delta_i\right) + g^1_{0012}\left(k_i^2\delta_i^2\right) + \\ &+ g^1_{2100}\left(-k_i\delta_i^{-1}\right) + g^1_{1101}\left(-k_i^2\right) \end{aligned} \tag{3.11}$$

$$\begin{aligned} P_2^i &= g^2_{2001}\left(-3k_i\right) + g^2_{1110}\left(k_i\right) + g^2_{1200}\left(-k_i^2\delta_i^{-1}\right) + \\ &+ g^2_{0201}\left(-k_i^3\right) + g^2_{0111}\left(k_i^2\delta_i\right) + g^2_{0003}\left(-3k_i^3\delta_i^2\right) + g^2_{1002}\left(-3k_i^2\delta_i\right) \\ G_1^{ij} &= g^1_{2100}\left(-2k_i\delta_j^{-1}\right) + g^1_{1101}\left(-2k_ik_j\right) + g^1_{0300}\left(-6k_ik_j^2\delta_j^{-1}\right) + \\ &+ g^1_{0210}\left(2k_j^2\delta_i\delta_j^{-1} + 4k_ik_j\right) + g^1_{0120}\left(-2k_i\delta_j - 4k_j\delta_i\right) + \\ &+ g^1_{1011}\left(2k_i\delta_i\right) + g^1_{0102}\left(-2k_ik_j^2\delta_j\right) + g^1_{0012}\left(2k_j^2\delta_i\delta_j\right) \\ G_2^{ij} &= g^2_{2001}\left(-2k_i\delta_i\delta_j^{-1} - 4k_j\right) + g^2_{1110}\left(2k_j\right) + g^2_{1200}\left(-2k_j^2\delta_1^{-1}\right) + \end{aligned}$$

$$+g_{0201}^2\left(-2k_i k_j^2 \delta_i \delta_j^{-1}\right)+g_{0111}^2\left(k_i k_j \delta_i\right)+g_{0003}^2\left(-6k_i k_j^2 \delta_i \delta_j\right)+g_{1002}^2\left(-2k_j^2 \delta_j - 4k_i k_j \delta_i\right)$$

apart from a positive constant.

Conditions (3.9)–(3.11) were verified using a computer: the system is asymptotically stable in domain 1 and unstable in domain 2.

We remark that the stability conditions (3.9)–(3.11) were not verified at any point (α, β) on the curve $\alpha\beta = 2$, on which the substitution (3.7) is undefined, or on the fourth-order resonance curve $\delta_i = 3\delta_2$, for which Kamenkov's criterion is inapplicable.

The following conclusions can be drawn on the basis of the theorem in Sec. 2. The unperturbed motion (1.8) corresponding to the uniform rotation of the body about an axis of symmetry orthogonal to the plane of the orbit is asymptotically stable with respect to ψ, $\psi^{\boldsymbol\cdot}$, θ, $\theta^{\boldsymbol\cdot}$, q_n, $q_n^{\boldsymbol\cdot}$ for (α, β) belonging to domain 1 (with the exception of the above-mentioned values of the parameters) and unstable for all remaining values (α, β).

The integral $\omega_3 = \text{const}$ implies asymptotic stability with respect to φ in domain 1.

Remarks. 1. To define stability in systems with an infinite number of degrees of freedom it is necessary to introduce a measure of deviation of the perturbed state from the unperturbed one. The norm

$$\|z\| = \sup_n(|z_1|, |z_2|, \ldots) \tag{3.12}$$

in the space of sequences $z = (z_1, \ldots, z_2, \ldots)$ has been used as such a measure [10, 11]. The asymptotic stability and instability obtained above should be understood, respectively, as the asymptotic stability or instability in the space of sequences $z = (x_1, x_2, x_1', x_2', \bar\xi_1, \bar\xi_1', \bar\xi_2, \bar\xi_2', \ldots)$ $(\bar\xi_n = A^{*1/2}\xi_n)$ with (3.12) as the measure of deviation (the corresponding definitions are stated in [14]).

It can be shown that similar conclusions also hold in the space of sequences with the norm

$$\|z\|^2 = x_1^2 + x_2^2 + x_1'^2 + x_2'^2 + \sum_{n=1}^{\infty} \bar\xi_n'^2 + \sum_{n=1}^{\infty} \omega_n^2 \bar\xi_n^2$$

which is the natural norm of the configuration space of the problem.

2. If there is no internal viscosity $(b = 0)$, the system under consideration has the generalized energy integral

$$H = \frac{1}{2}(\boldsymbol\omega \cdot J\boldsymbol\omega) + \frac{1}{2}\sum_{n=1}^{\infty} q_n^{\boldsymbol\cdot 2} + \Pi_* + \frac{\omega_0^2}{2}(3\boldsymbol\gamma \cdot J\boldsymbol\gamma - \boldsymbol\beta \cdot J\boldsymbol\beta \cdot 2A - C)$$

where $\boldsymbol\beta$ is the unit vector in the binormal direction to the orbit.

The second variation of H in the neighbourhood of the stable motion (1.8) can be represented in the (dimensionless) form

$$\delta^2 H = \frac{1}{2}\left(x_1'^2 + x_2'^2 + (\alpha\beta - 1)x_1^2 + (\alpha\beta + 3\alpha - 4)x_2^2 + \sum_{n=1}^{\infty} \bar\xi_n'^2 + \sum_{n=1}^{\infty}\left(\omega_n^2 + 1\right)\bar\xi_n^2\right)$$

In the case of a viscoelastic body

$$H^{\boldsymbol\cdot} = -2\Psi \leqslant 0$$

Using the second Lyapunov method, one can find [15] that the unperturbed motion is stable with respect to x_1, x_1', x_2, x_2', ξ_n and asymptotically stable with respect to ξ_n' in domain 1

$$(\alpha\beta - 1) > 0, \quad (\alpha\beta + 3\alpha - 4) > 0$$

Using this approach, one is unable to draw any conclusions concerning asymptotic stability with respect to all the variables or instability.

The above discussion shows the merits of the proposed approach to studying stability in the problem under consideration.

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